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Jake Weber

*University of Northern Iowa*

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An Exploration in Ramsey Theory

An Abstract of a Thesis  
Submitted  
in Partial Fulfillment  
of the Requirement for the Degree  
Master of Arts

Jake Weber  
University of Northern Iowa  
May 8, 2020

## ABSTRACT

We present several introductory results in the realm of Ramsey Theory, a subfield of Combinatorics and Graph Theory. The proofs in this thesis revolve around identifying substructure amidst chaos. After showing the existence of Ramsey numbers of two types, we exhibit how these two numbers are related. Shifting our focus to one of the Ramsey number types, we provide an argument that establishes the exact Ramsey number for  $h(k, 3)$  for  $k \geq 3$ ; this result is the highlight of this thesis. We conclude with facts that begin to establish lower bounds on these types of Ramsey numbers for graphs requiring more substructure.

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This Study by: Jake Weber

Entitled: An Exploration in Ramsey Theory

Has been approved as meeting the thesis requirement for the  
Degree of Master of Arts.

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Date

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Dr. Adrienne Stanley, Chair, Thesis Committee

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Date

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Dr. Douglas Shaw, Thesis Committee Member

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Date

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Dr. Douglas Mupasiri, Thesis Committee Member

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Date

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Dr. Jennifer Waldron, Dean, Graduate College

*I would like to dedicate this thesis to Dr. Adrienne Stanley. Dr. Stanley has been my instructor, research advisor, thesis committee chair, and greatest academic supporter over the last five years of my career. She has encouraged me in my pursuit of a greater understanding of the world of mathematics, inspiring me to find my own niche within the field. She exposed me to different perspectives on teaching and how to learn best myself, helping to deepen my understanding of content. Much of what I have learned from her has gotten me to where I am today. Thank you for your unwavering support throughout it all.*

*Woot!*

## ACKNOWLEDGEMENTS

First, I am inclined to express my gratitude to the faculty that have instilled inspiration and diligence for the study and teaching of mathematics in me. These individuals have supported me and shown me purpose in a life steeped in mathematics: Dr. Catherine Miller, Dr. Douglas Mupasiri, Dr. Michael Prophet, Dr. Douglas Shaw, and Dr. Adrienne Stanley.

I would like to thank the members of my thesis committee: Dr. Adrienne Stanley, Dr. Douglas Shaw, and Dr. Douglas Mupasiri. They have given their time and expertise to ensure a worthwhile product is constructed. Thank you for your contribution to my success at the University of Northern Iowa.

I must acknowledge my family: Patrick, Julie, Maggie, and Cole. These individuals have always supported my dreams and aspirations, lending me their ear whenever I needed a partner to bounce ideas and thoughts off of.

My thanks and appreciation to the cohorts of mathematics students I've had the pleasure of working with during my time at the University of Northern Iowa. Mathematics, like other things in life, is more fun when working with a team, dare I say family.

Lastly, to the people who made UNI feel like home; the people who gave me purpose outside of the classroom; the people who knew how to make me smile; thank you.

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## CHAPTER 0

### Introduction

Often as individuals, we come across situations where we wonder what it would take to guarantee a certain outcome. Just as often, mathematicians ask the same type of question. By asking such a question, we are asserting that there is some sort of underlying substructure that can explain when an outcome occurs.

Mathematicians love to research and uncover substructure. The problems addressed in this thesis are no different in that we look for substructure within a graph.

Graphs are extremely useful tools. A graph is a collection of vertices and edges. These vertices sometimes can be thought of as objects with the edges between them indicating a relationship between the two. This characterization of a graph might sound quite familiar. In fact, informally throughout our lives, we have been familiarizing ourselves with graphs. For instance, a map is made up of cities (vertices) that have roads between them (edges describing the ability to travel from a given city to another). When in a maze, one stands at a vertex and chooses a direction to walk, selecting which edge should be taken. Or even cooking a meal can be seen as a graph event; each vertex could represent a state of the dinner (what is ready to eat and what is not), and each edge could represent an action of preparing something for the meal; there are often many orders in which things can be completed in order to finish the dinner.

Problems involving graphs characterized in this way can be traced back to 1735; the branch of mathematics called Graph Theory originated with the Königsburg Bridge problem. Königsburg was a thriving city located along the Pregel River in Prussia, and the city was partitioned into four main land masses by the river. The four regions of the city were connected by a series of seven bridges, and it was common for a citizen of Königsburg to ask if they could go for a walk and cross each bridge exactly once. Eventually, Leonard Euler, a famous and prolific mathematician, had the problem proposed to him.

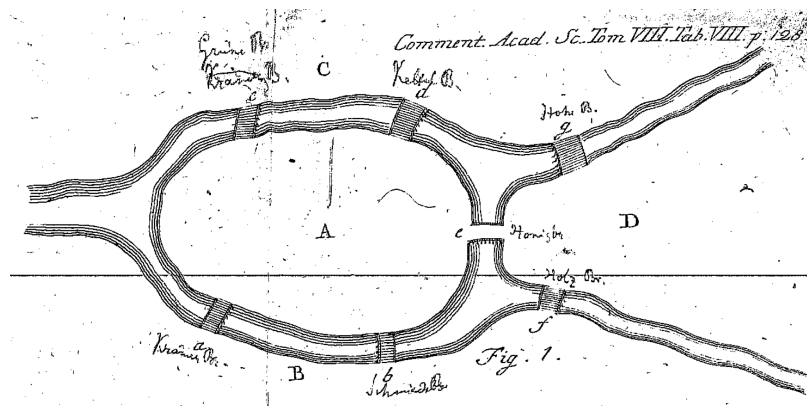


Figure 1

Euler, after some thought, concluded that there was no way to cross each bridge exactly once. He solved the problem by determining how many times (8) the regions of Königsburg would be visited if only crossing each bridge once, and he determined how many times each individual region must be visited based off of the number of bridges that lead to the given region. Each viewpoint led to a different number of regions visited, and so it was not possible to cross each bridge exactly once. The regions and bridges of Königsburg are shown in fig. 1 [8] .

After another 190 years or so, giving graph theory some time to develop and stand on its own, a question that kicked off my thesis inquiry was posed. Frank Ramsey, a young British mathematician, proved that graphs with large enough vertex sets guarantee an induced subgraph that is either complete or independent.

One can phrase a special case of his conclusion as such. If nine people are gathered into a room, Ramsey guaranteed that some four of the nine will all be familiar with each other or some three of the nine will all be unfamiliar with each other. In our math phrasing, each person translates to a vertex, and each type of relationship (familiar or unfamiliar) corresponds to an edge, or lack there of.

In the thesis to follow, we consider these questions of Ramsey and focus on a related question; instead of a complete or independent induced subgraph being guaranteed,

can one guarantee a cycle subgraph or an independent induced subgraph? A special case of a conclusion of this form can be phrased like so. If Pat, Julie, Maggie, Jake, Cole, Lou, and Mary Ann are in a room together, we can guarantee that some four of the seven, say Pat; Julie; Maggie; and Cole, will be familiar in the following way: Pat knows Julie, and Julie knows Maggie, and Maggie knows Cole, who knows Pat. It is similar to being four degrees of separation away from someone else who knows you. Otherwise, if this chain of familiarity is not present, some three of the seven will all be unfamiliar with each other.

With this cute party anecdote in the back of our minds, let us dive into the representations and language that describe these underlying graphs, subgraphs, and relationships.

## CHAPTER 1

### Language and Conventions

Let us begin by laying a foundation for effective and precise communication. The language established in this chapter will be valuable in understanding the expression of thought found within. With no intent of reinventing the wheel, we use definitions and notation consistent with Brualdi [3].

We start the definitions section by first defining what a graph is, its components (vertices and edges), and how these components may relate to each other.

**Definition 1.1.** A **graph**  $G$  is composed of two types of objects. It has a finite set  $V = \{x_0, x_1, x_2, \dots, x_{k-1}\}$  of elements called **vertices** and a set  $E$  of pairs of distinct vertices called **edges**. We denote the graph whose vertex set is  $V$  and whose edge set is  $E$  by  $G = (V, E)$ . When more than one graph is present, we denote the the vertex set of  $G$  as  $V(G)$  and the edge set of  $G$  as  $E(G)$ .

**Definition 1.2.** The number  $n$  of vertices in the set  $V$  is called the **order** of the graph  $G$ .

**Definition 1.3.** If  $\alpha = \{x, y\} = \{y, x\}$  is an edge of graph  $E$ , then we say that  $\alpha$  **joins**  $x$  and  $y$  and that  $x$  and  $y$  are **adjacent**. We also say that  $x$  and  $\alpha$  are **incident**, and  $y$  and  $\alpha$  are **incident**. Lastly, we refer to  $x$  and  $y$  as the vertices of the edge  $\alpha$ .

**Definition 1.4.** The **degree** of vertex  $x$  in a graph  $G$  is the number  $\deg(x)$  of edges that are incident with  $x$ .

**Definition 1.5.** The **total degree** of a graph  $G$  is the sum of the degrees of all vertices of  $G$ .

In our main results, we often consider a graph and look for some type of substructure. If some substructure is present, then it is observable in a subset of vertices and edges of the given graph. Next we define subgraph and induced subgraph; each gives

us a different lens through which to view a given graph. An induced subgraph preserves the most information from a given graph where a subgraph need not.

**Definition 1.6.** Let  $G = (V, E)$  be a graph. Let  $U$  be a subset of  $V$  and  $F$  a subset of  $E$  such that the vertices of each edge in  $F$  belong to  $U$ . We call the graph  $S = (U, F)$  a **subgraph** of  $G$ . Notice, some edges of  $G$  might be omitted in the subgraph  $S$ .

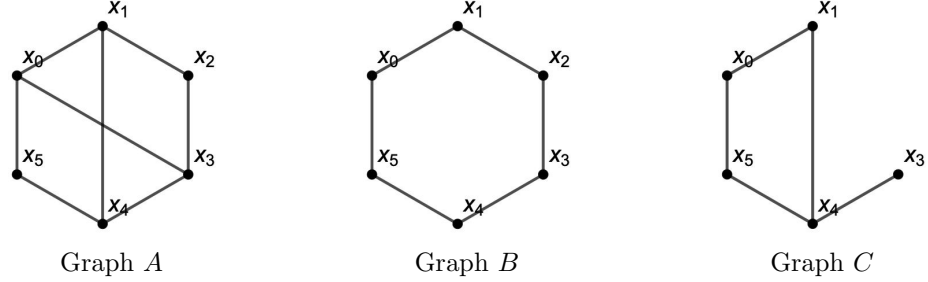


Figure 1.1

In fig. 1.1,  $B$  and  $C$  are subgraphs of  $A$ . Notice,  $B$  omits edges  $\{x_0, x_3\}$  and  $\{x_1, x_4\}$ . Similarly,  $C$  omits edges  $\{x_0, x_3\}$ ,  $\{x_1, x_2\}$ , and  $\{x_2, x_3\}$ .

**Definition 1.7.** Let  $G = (V, E)$  be a graph. Let  $S = (U, F)$  be a subgraph of  $G$  such that  $F$  consists of all edges of  $G$  that join vertices in  $U$ . We call  $S$  an **induced subgraph** of  $G$ . Here no edges of  $G$  are omitted in the subgraph  $S$ .



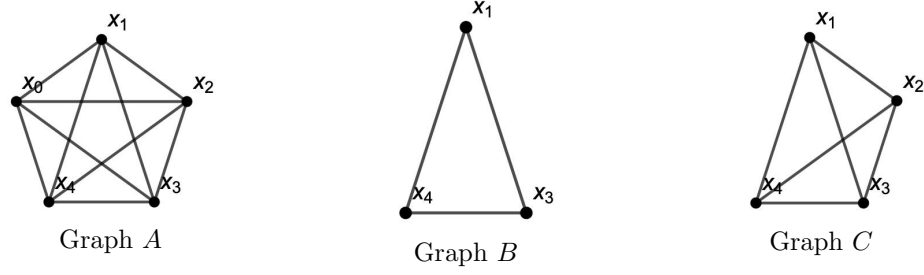


Figure 1.2

In fig. 1.2,  $B$  and  $C$  are induced subgraphs of  $A$ . Notice,  $B$  includes all edges connecting vertices  $x_1, x_3$ , and  $x_4$ . Similarly,  $C$  includes all edges connecting vertices  $x_1, x_2, x_3$ , and  $x_4$ . Additionally,  $B$  and  $C$  are subgraphs of  $A$  as all induced subgraphs are subgraphs.

To reiterate, subgraphs are important because they help us understand how the larger graph is built so to speak. We often consider subgraphs with particular structures. Before diving into specific definitions, let us informally describe some of these useful and interesting graphs/structures.

We are interested in disjoint graphs. Vertices of these graphs can be separated into groups such that there are no edges connecting vertices from different groups. We also talk about independent graphs; vertices of these graphs have no edges between them.

Not only do we focus on graphs that have some sense of separateness, but we look at graphs that have some notion of togetherness. In a walk, we describe a sequence of edges that connect a beginning vertex to an ending vertex, and we further develop this idea of a walk by defining a cycle. A cycle is exactly what you would expect; it is a sequence of edges that connects a sequence of vertices, starting and ending at the same vertex. Lastly, complete graphs are those in which all vertices are pairwise adjacent.

In this thesis, we analyze the underlying structure of graphs with respect to the following types of subgraphs: disjoint, independent, cycle, and complete.

**Definition 1.8.** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs. We say  $G_1$  and  $G_2$  are **disjoint** if  $V_1 \cap V_2 = \emptyset$  and for any vertex  $x_1 \in V_1$  and any vertex  $x_2 \in V_2$ , no edge joins  $x_1$  and  $x_2$ .

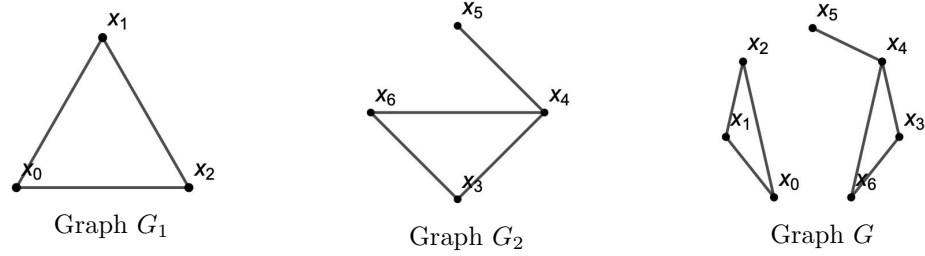


Figure 1.3

From fig. 1.3, we say  $G_1$  and  $G_2$  are disjoint. Often, we consider a pair of disjoint graphs that are subgraphs of a larger graph. To illustrate this point, notice  $G$  has both  $G_1$  and  $G_2$  for subgraphs.

**Definition 1.9.** Let  $G = (V, E)$  be a graph. A sequence of  $k$  edges of the form

$$\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{k-1}, x_k\}$$

is called a **walk** of length  $k$ , and this walk joins the vertices  $x_0$  and  $x_k$ . We also denote the walk by

$$x_0 - x_1 - x_2 - \dots - x_k.$$

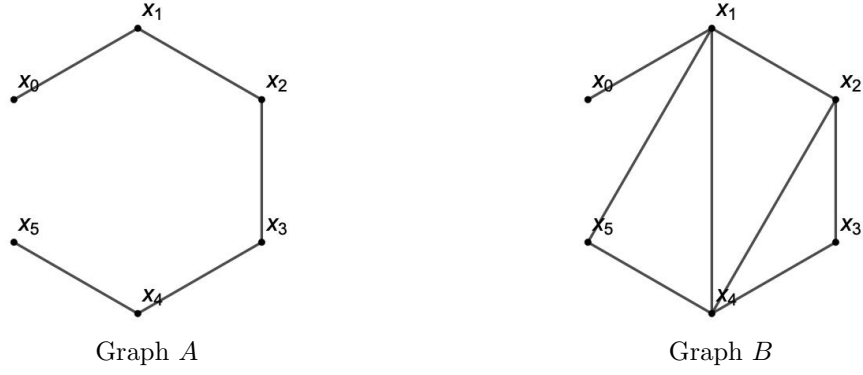


Figure 1.4

In fig. 1.4,  $x_0 - x_1 - x_2 - x_3 - x_4 - x_5$  is an example of a walk of length 5 found in  $A$ . As well,  $x_0 - x_1 - x_2 - x_3 - x_4 - x_1 - x_5 - x_4 - x_2$  is an example of a walk of length 8 found in  $B$ .

**Definition 1.10.** A **cycle** of order  $k$  is a graph whose edges form a walk with the following properties.

1. The length of this walk is  $k$ .
2. If  $x_0, x_1, x_2, \dots, x_{k-1}$  are the vertices of this walk with  $x_0$  and  $x_{k-1}$  being the respective beginning and ending vertices, then  $x_0 = x_{k-1}$ .
3. Other than the beginning and ending vertices of this walk, every other vertex is distinct.

Thus a cycle can be denoted by

$$x_0 - x_1 - x_2 - \dots - x_{k-1} - x_0.$$

We denote a cycle with  $k$  vertices by  $C_k$ .

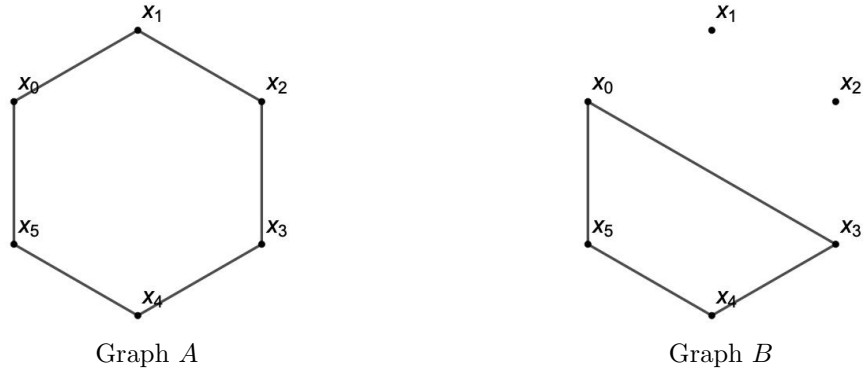
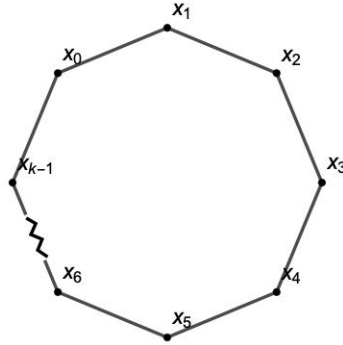


Figure 1.5

In fig. 1.5,  $x_0 - x_1 - x_2 - x_3 - x_4 - x_5 - x_0$  is an example of a cycle of length 6 found in  $A$ . In addition,  $x_0 - x_3 - x_4 - x_5 - x_0$  is an example a cycle of length 4 found in  $B$ . Notice this cycle is an induced subgraph of  $B$ .

**Convention 1.11.** If  $x_0, x_1, \dots, x_{k-1}$  are the distinct vertices in a cycle, then  $x_{i-1}$  and  $x_i$  are adjacent for  $i \in \{1, \dots, k-1\}$  as well as  $x_{k-1}$  and  $x_0$ .

Figure 1.6: Cycle  $C_k$ 

Observe in fig. 1.6,  $x_{i-1}$  and  $x_i$  are adjacent for  $i \in \{1, \dots, k-1\}$  as well as  $x_{k-1}$  and  $x_0$ .

The line break between  $x_6$  and  $x_{k-1}$  denotes the continuance of the cycle/walk from vertex  $x_6$  to vertex  $x_{k-1}$ .

**Definition 1.12.** A graph of order  $n$  is called **independent**, provided no two of its vertices are adjacent. We denote such a graph by  $I_n$ .

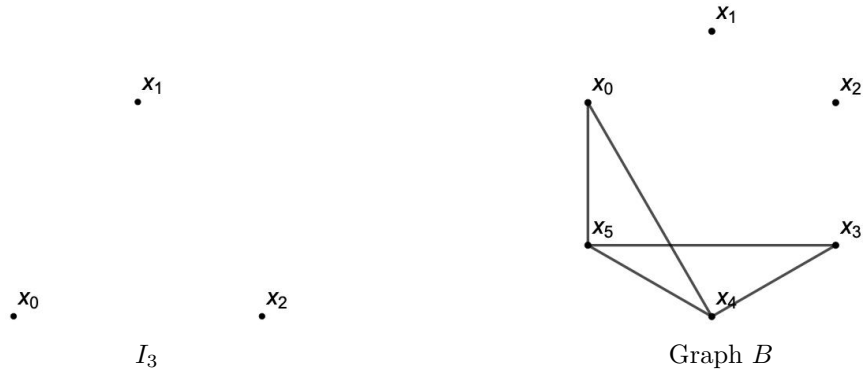


Figure 1.7

In fig. 1.7, the left graph is an independent graph of order 3. In  $B$ ,  $x_0, x_1, x_2$ , and  $x_3$  are the vertices of an independent graph of order 4. Notice this independent graph is an induced subgraph of  $B$ .

**Definition 1.13.** A graph of order  $n$  is called **complete**, provided each pair of distinct vertices is adjacent. Thus in a complete graph each vertex is adjacent to every other vertex. A complete graph of order  $n$  is denoted by  $K_n$ .

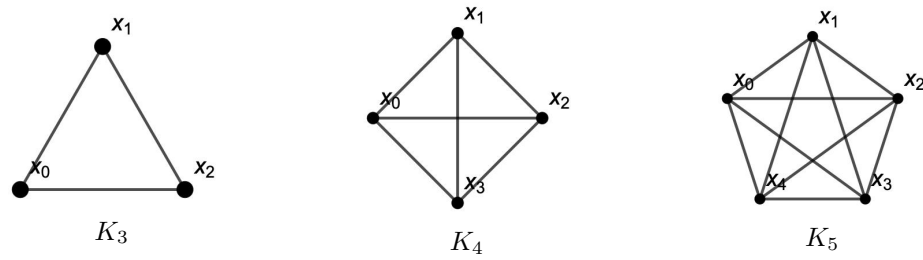


Figure 1.8

In fig. 1.8, complete graphs of order 3, 4, and 5 are shown.

**Example 1.14.**

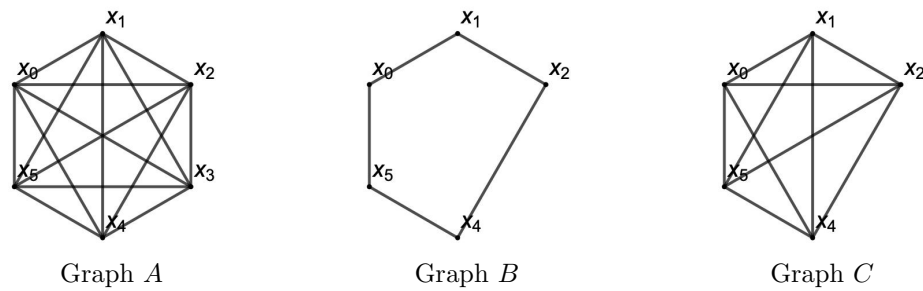


Figure 1.9

Later on, we focus on cycles and complete graphs. Motivated by definition 1.17 and definition 1.18, we look for cycles and complete graphs that are subgraphs and induced subgraphs. In fig. 1.9,  $B$  is a cycle of order 5, and it is a subgraph of  $A$ . Alternatively,  $C$  is a complete graph of order 5, and it is an induced subgraph of  $A$ .

**Convention 1.15.** *There are two styles of graphs used in the figures throughout this*

paper, and their difference regards how we present our known information. In both styles, solid colored edges denote two adjacent vertices. The styles are:

1. If a graph does not use dotted edges, then vertices not connected by an edge are known to be not adjacent.
2. If a graph uses dotted edges, then the vertices connected by a dotted edge are not adjacent, and if vertices are not connected by an edge, we make no assumptions about their adjacency.

We employ the use of both styles in this paper because they help us communicate in two types of ways. Style 1 is used in the scenario when we know all relationships between each pair of vertices. Style 2 is used when we do not know about the adjacency of each pair of vertices; this style is used more often when we know a property of some graph and construct it more fully through deduction.

**Definition 1.16.** A graph  $G^C$  is said to be the **complement** of graph  $G$  if:

1.  $V(G^C) = V(G)$ .
2. Two vertices of  $G^C$  are adjacent if and only if they are not adjacent in  $G$ .

**Definition 1.17** (Ramsey's Theorem/numbers). Let  $k, l \in \mathbb{N}$  such that  $k \geq 2$  and  $l \geq 2$ . Let  $r \in \mathbb{N}$  be the least integer such that if  $G$  is a graph with  $|V(G)| = r$ , then  $G$  has an induced subgraph  $K_k$  or an induced subgraph  $I_l$ . We call  $r$  a Ramsey number and denote it as a function of  $k$  and  $l$  by  $f(k, l) = r$ .

We recognize that we have not shown the existence of such a number  $r$ , but its existence is well known [33]. We will prove there exists a finite upper bound on  $r$  in theorem 2.5.

Ramsey proved both an infinite and finite version of his theorem. In this paper, we focus on the finite version. In fact, since Ramsey's proof of the existence of such a number  $r$  in 1928, only nine Ramsey numbers have been discovered. The most recent discovery

was from McKay and Radziszowski [31], proving  $f(4, 5) = 25$ . Even though many Ramsey numbers are not explicitly known, progress has been made in providing upper and lower bounds for many Ramsey numbers.

Instead of continuing to hunt down Ramsey numbers, we introduce a related number which will be the main topic of this thesis.

**Definition 1.18.** *Let  $k, l \in \mathbb{N}$  such that  $k \geq 2$  and  $l \geq 2$ . Let  $s \in \mathbb{N}$  be the least integer such that if  $G$  is a graph with  $|V(G)| = s$ , then  $G$  has a subgraph  $C_k$  or an induced subgraph  $I_l$ . We denote  $s$  as a function of  $k$  and  $l$  by  $h(k, l) = s$ .*

We would like to point out that this related definition sets the stage for the main theorems of this thesis. Moving forward, our main goal is to determine  $h(k, l)$ . We are successful in identifying the value of  $h(k, l)$  for  $l = 3$  and  $k \geq 3$ . We find a lower bound for the value of  $h(k, 3)$  for  $k \geq 3$  by constructing a sequence of counterexamples. We show that this lower bound is indeed an optimal lower bound by an induction proof. Thus, we have determined the value for  $h(k, 3)$ .

Besides focusing on  $h(k, l)$ , we explore the relationships between the values of  $h(k, l)$  and  $f(k, l)$ . It is important for us to connect our Ramsey-like question, regarding the value of  $h(k, l)$ , to the original inquiry; as one can imagine, knowing the value of  $f(k, l)$  or  $h(k, l)$  assists in finding the other. Let us formally begin our exploration.



## CHAPTER 2

### Main Results

#### 2.1 Initial Values of $f(k, l)$

Let us get our feet wet, so to speak, within Ramsey Theory. We will begin by finding the explicit values of two Ramsey numbers:  $f(3, 3)$  and  $f(4, 3)$ . These types of proofs require two parts. The first part provides an optimal graph such that the required criteria is not satisfied. The second part proves that the required criteria is satisfied for graphs with exactly one more vertex than the previously provided graph.

**Lemma 2.1.**  $f(3, 3) > 5$ .

Proof. Let  $G$  be a graph as in fig. 2.1. Observe,  $G$  does not have an induced  $K_3$ . (If  $G$  did, a black triangle would be visible.)

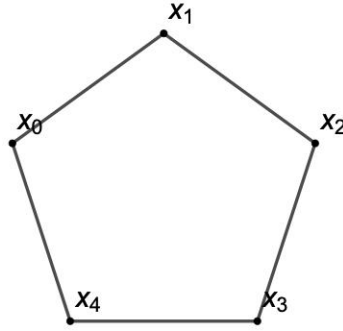


Figure 2.1:  $f(3, 3)$  counterexample

Without loss of generality, let us consider  $x_0$ . If  $x_0$  were a vertex of an  $I_3$ ,  $x_2$  and  $x_3$  would have to be the remaining two vertices in the  $I_3$  as  $x_0$  is adjacent to  $x_1$  and  $x_4$ . However,  $x_2$  and  $x_3$  are adjacent; thus no vertex of  $G$  is a vertex of an  $I_3$ .

Thus  $G$  has neither an induced  $K_3$  nor  $I_3$ . □

**Lemma 2.2.**  $f(3, 3) = 6$ .

Proof. Let  $G$  be a graph such that  $|V(G)| = 6$  and  $V(G) = \{x_0, \dots, x_5\}$ . Suppose  $G$  does not have an induced  $I_3$ . We will show  $G$  has an induced  $K_3$ .

Let us consider  $x_0$ .

Case 1: Suppose  $x_0$  is disjoint from  $S$  where  $S$  is an induced subgraph of  $G$  and  $|V(S)| \geq 3$ . Let  $S$  be such a graph. Since  $G$  does not have an induced  $I_3$  and  $x_0$  is disjoint from  $S$ , every pair of vertices of  $S$  must be adjacent. Thus  $S$  is a complete graph of order at least 3. Thus  $S$  has an induced  $K_3$ ; hence  $G$  has an induced  $K_3$ .

Case 2: Suppose  $x_0$  is not adjacent with at most 2 vertices of  $G$ . Thus,  $x_0$  is adjacent with at least 3 vertices of  $G$ . Let  $x_0$  be adjacent with the vertices of  $S$  where  $|V(S)| \geq 3$  and  $S$  is an induced subgraph of  $G$ . Since  $G$  does not have an induced  $I_3$ , without loss of generality let  $x_1, x_2 \in V(S)$  be adjacent. Notice  $\{x_0, x_1, x_2\}$  are the vertices of an induced  $K_3$ . Thus  $G$  has an induced  $K_3$ .

Thus  $f(3, 3) \leq 6$ . From lemma 2.1, we know  $f(3, 3) > 5$ . Using both of these facts, we have  $f(3, 3) = 6$ . □

**Lemma 2.3.**  $f(3, 4) > 8$ .

Proof. Let  $G$  be a graph as in fig. 2.2. We provide an argument for why  $G$  has neither an induced  $K_3$  nor an induced  $I_4$ . First, observe  $G$  does not have an induced  $K_3$ .

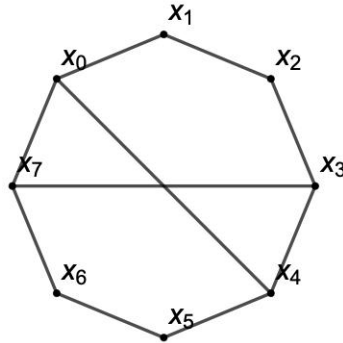


Figure 2.2:  $f(4, 3)$  counterexample

We will now show  $G$  does not have an induced  $I_4$ . We shall consider vertices of

different degree separately.

Case 1: Without loss of generality, let us consider vertex  $x_0$ . By way of contradiction, suppose  $x_0$  is a vertex of an  $I_4$  called  $I$ . Then  $x_1, x_4, x_7 \notin V(I)$ . Since  $x_2$  and  $x_3$  are adjacent, no more than one of them is in  $V(I)$ . Similarly,  $x_5$  and  $x_6$  are not both in  $V(I)$ . Since three more vertices are in  $I$ , either  $x_5$  and  $x_6$  are in  $V(I)$  or  $x_1$  and  $x_2$  are in  $V(I)$ .  $\nmid$

Case 2: Without loss of generality, let us consider vertex  $x_1$ . By way of contradiction, suppose  $x_1$  is a vertex of an  $I_4$  called  $I$ . Then  $x_0, x_2 \notin V(I)$ . Since  $x_3 - x_4 - x_5 - x_6 - x_7$  is a walk, no two consecutive vertices are in  $V(I)$ . Thus,  $x_3, x_5, x_7 \in V(I)$ . However,  $x_3$  and  $x_7$  are adjacent.  $\nmid$

Thus  $G$  does not have an induced  $I_4$ . □

**Lemma 2.4.**  $f(3, 4) = 9$ .

Proof. Let  $G$  be a graph such that  $|V(G)| = 9$  and  $V(G) = \{x_0, \dots, x_8\}$ . Suppose  $G$  does not have an induced  $K_3$ . We will show  $G$  has an induced  $I_4$ .

Let us consider  $x_0$ .

Case 1: Suppose  $x_0$  is not adjacent to at least six other vertices of  $G$ . By lemma 2.2, within that subgraph,  $S$ , of at least six vertices, there exists an induced  $K_3$  or an induced  $I_3$ . Since  $G$  does not have an induced  $K_3$ ,  $S$  does not have an induced  $K_3$ . Thus,  $S$  has an induced  $I_3$ . Since  $x_0$  is not adjacent to each vertex of  $S$  and  $S$  has an induced  $I_3$ , observe  $V(I_3) \cup \{x_0\}$  are the vertices of an induced  $I_4$ . Thus  $G$  has an induced  $I_4$ .

So  $x_0$  is not adjacent to at most five other vertices of  $G$ . This means that  $x_0$  is adjacent to at least three other vertices of  $G$ .

Case 2: Suppose  $x_0$  is not adjacent to at most four vertices of  $G$ . Then,  $x_0$  is adjacent to at least four vertices of  $G$ . Thus we can let  $S$  be an induced subgraph of  $G$  such that  $x_0$  is adjacent to each vertex of  $S$  and  $|V(S)| \geq 4$ . Since  $G$  does not have an induced  $K_3$  and  $x_0$  is adjacent to each vertex of  $S$ , every pair of vertices of  $S$  must not be adjacent. Thus  $S$  has an induced  $I_4$ ; hence  $G$  has an induced  $I_4$ .

Case 3: Suppose  $x_0$  is not adjacent to exactly five vertices of  $G$ . Since  $x_0$  is any generic vertex of  $G$ , this is the case in which each vertex of  $G$  is not adjacent to exactly five vertices of  $G$ . Thus each vertex of  $G$  is adjacent to exactly three vertices of  $G$ . Then, the total degree of  $G$  is 27. However, for each edge, two is added to the total degree of the graph (one to each of the degrees of its two vertices). Thus a graph with an odd total degree does not exist.  $\nexists$  □

We begin with proofs of some of the smallest Ramsey numbers because they help us get a feeling for the type of arguments that will be made later on. As well, these two proofs will be referenced again as they act as building blocks for proving structure in other graphs.

## 2.2 The $f(k, l)$ and $h(k, l)$ Relationships

Now that we have shown that Ramsey's Theorem holds for two  $(k, l)$  pairs, we will provide a constructive proof of Ramsey's Theorem that is true for all  $k, l \in \mathbb{N}$  such that  $k, l \geq 2$ . As well, we will show our Ramsey-like number,  $h(k, l)$ , also exists as a consequence of Ramsey's Theorem.

**Theorem 2.5** (Ramsey's Theorem). *Let  $k, l \in \mathbb{N}$  such that  $k, l \geq 2$ . Then,  $f(k, l) < \infty$ . In other words,  $f(k, l)$  exists.*

Proof. Let  $k, l \in \mathbb{N}$  such that  $k, l \geq 2$ . We will show  $f(k, l)$  exists by double induction on  $k$  and  $l$ . Not only will we prove such a number exists, but we will provide an upper bound for  $f(k, l)$  with  $k, l \geq 3$ .

Base Case: We will show  $f(2, l) < \infty$  and  $f(k, 2) < \infty$  for  $k, l \in \mathbb{N} \setminus \{1\}$ .

Let  $G$  be a graph with  $|V(G)| = l$ . Suppose  $G$  does not have an induced  $I_l$ . We will show  $G$  has an induced  $K_2$ . Since  $G$  does not have an induced  $I_l$ , there exists two vertices of  $G$  that are adjacent. Thus  $G$  has an induced  $K_2$ .

Thus  $f(2, l) \leq l$ .

Similarly,  $f(k, 2) \leq k$ .

Let us proceed with our induction on  $k$  and  $l$ . Now, suppose  $k, l \geq 2$ . As our inductive hypothesis, let  $f(k, l+1) < \infty$  and  $f(k+1, l) < \infty$ . We will show  $f(k+1, l+1) < \infty$ . Even better, we will establish  $f(k+1, l+1) \leq f(k, l+1) + f(k+1, l)$ .

Let  $G$  be a graph such that  $|V(G)| = f(k, l+1) + f(k+1, l)$ . Fix  $x_0$  and  $x_1$  in  $V(G)$ . Let

$$V(S) = \{y \in V(G) : y \text{ is adjacent to } x_0, y \neq x_0, x_1\}$$

where  $S$  is the induced subgraph of  $G$  with this vertex set. Similarly, let

$$V(T) = \{z \in V(G) : z \text{ is not adjacent to } x_0, z \neq x_0, x_1\}$$

where  $T$  is the induced subgraph of  $G$  with this vertex set.

If  $|V(S)| \geq f(k, l+1)$ , then by the inductive hypothesis,  $S$  has an induced  $K_k$  or an induced  $I_{l+1}$ . If  $S$  has an induced  $I_{l+1}$ , then we are finished as  $G$  would have an induced  $I_{l+1}$ . Suppose  $S$  has an induced  $K_k$ . Since  $x_0$  is adjacent to every vertex of  $S$ ,  $V(K_k) \cup \{x_0\}$  are the vertices of a  $K_{k+1}$ . Thus  $G$  would have an induced  $K_{k+1}$ . So, suppose  $|V(S)| \leq f(k, l+1) - 1$ .

If  $|V(T)| \geq f(k+1, l)$ , then by the inductive hypothesis,  $T$  has an induced  $K_{k+1}$  or an induced  $I_l$ . If  $T$  has an induced  $K_{k+1}$ , then we are finished as  $G$  would have an induced  $K_{k+1}$ . Suppose  $T$  has an induced  $I_l$ . Since  $x_0$  is not adjacent to every vertex of  $T$ ,  $V(I_l) \cup \{x_0\}$  are the vertices of an  $I_{l+1}$ . Thus  $G$  would have an induced  $I_{l+1}$ . So, suppose  $|V(T)| \leq f(k+1, l) - 1$ .

Now, notice

$$\begin{aligned} f(k, l+1) + f(k+1, l) &= |V(G)| \\ &= |V(S)| + |V(T)| + |\{x_0, x_1\}| \\ &\leq [f(k, l+1) - 1] + [f(k+1, l) - 1] + 2 \\ &= f(k, l+1) + f(k+1, l). \end{aligned}$$

Thus we have  $|V(S)| = f(k, l + 1) - 1$  and  $|V(T)| = f(k + 1, l) - 1$  as shown in fig. 2.3.

Now,  $x_0$  can be adjacent or not adjacent with  $x_1$ .

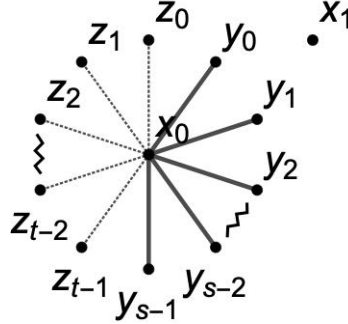


Figure 2.3: Note,  $|V(S)| = s$ , and  $|V(T)| = t$ .

Case 1: Let  $x_1$  be adjacent to  $x_0$ . Consider the graph  $S'$  where  $V(S') = V(S) \cup \{x_1\}$ . So,  $|V(S')| = f(k, l + 1)$ . Thus, by the inductive hypothesis,  $S'$  has an induced  $K_k$  or an induced  $I_{l+1}$ . If  $S'$  has an induced  $I_{l+1}$ , then so does  $G$ . Suppose  $S'$  has an induced  $K_k$  called  $K$ . Since all vertices of  $S'$  are adjacent with  $x_0$ ,  $V(K) \cup \{x_0\}$  are the vertices of an induced  $K_{k+1}$ . Hence  $G$  has an induced  $K_{k+1}$ .

Case 2: Let  $x_1$  not be adjacent to  $x_0$ . Consider the graph  $T'$  where  $V(T') = V(T) \cup \{x_1\}$ . So,  $|V(T')| = f(k + 1, l)$ . Thus, by the inductive hypothesis,  $T'$  has an induced  $K_{k+1}$  or an induced  $I_l$ . If  $T'$  has an induced  $K_{k+1}$ , then so does  $G$ . Suppose  $T'$  has an induced  $I_l$  called  $I$ . Since all vertices of  $T'$  are not adjacent to  $x_0$ ,  $V(I) \cup \{x_0\}$  are the vertices of an induced  $I_{l+1}$ . Hence  $G$  has an induced  $I_{l+1}$ .

Thus we have shown that  $G$  is guaranteed to have an induced  $K_{k+1}$  or an induced  $I_{l+1}$ . Hence,  $f(k + 1, l + 1) \leq f(k, l + 1) + f(k + 1, l) < \infty$ .  $\square$

**Lemma 2.6.** *Let  $k, l \in \mathbb{N}$  such that  $k, l \geq 2$ . Then  $h(k, l) \leq f(k, l)$ .*

Proof. Let  $k, l \in \mathbb{N}$ . Let  $G$  be a graph such that  $|V(G)| = f(k, l)$ . Then,  $G$  has an induced  $K_k$  or an induced  $I_l$ . If  $G$  has an induced  $K_k$ , then clearly  $G$  has a  $C_k$  subgraph.

Otherwise,  $G$  has an induced  $I_l$ . Thus whenever  $|V(G)| = f(k, l)$ ,  $G$  is guaranteed to have

a  $C_k$  subgraph or an induced  $I_l$ .

Thus  $h(k, l) \leq f(k, l)$ .  $\square$

**Corollary 2.7.** *Let  $k, l \in \mathbb{N}$  such that  $k, l \geq 2$ . Then,  $h(k, l) < \infty$ .*

Proof. Let  $k, l \in \mathbb{N}$  such that  $k, l \geq 2$ . From theorem 2.5, we know  $f(k, l) < \infty$ . From lemma 2.6, we know  $h(k, l) \leq f(k, l)$ . Using both of these facts, we have  $h(k, l) < \infty$ .  $\square$

Now that we have proven that  $f(k, l)$  and  $h(k, l)$  exist for  $k, l \geq 2$ , we can proceed to talk about other characteristics that describe how the two functions are related.

This next relationship is a natural question, and even though we only speak about  $f(k, l)$  here, we later ask this question for  $h(k, l)$ . Since the complement of a complete graph is an independent graph, we derive our next equality regarding  $f(k, l)$ . This equality becomes insightful to us in lemma 3.1 when trying to determine an unknown  $h(k, l)$  value.

**Theorem 2.8.** *Let  $k, l \in \mathbb{N}$ . Then  $f(k, l) = f(l, k)$ .*

Proof. Let  $k, l \in \mathbb{N}$ . By Ramsey's Theorem (theorem 2.5),  $f(k, l), f(l, k) < \infty$ . Without loss of generality, let  $f(k, l) \leq f(l, k)$ . We will show  $f(k, l) = f(l, k)$  by showing  $f(l, k) \leq f(k, l)$ .

Let  $G$  be a graph such that  $|V(G)| = r = f(k, l)$ . We will show  $G$  has an induced  $K_l$  or an induced  $I_k$ . Consider  $G^C$  where  $G^C$  is the complement of  $G$ . Since  $|V(G^C)| = r = f(k, l)$ ,  $G^C$  has an induced  $K_k$  or an induced  $I_l$ .

Case 1: Suppose  $G^C$  has an induced  $K_k$ . Then  $G$  has an induced  $I_k$ . Thus  $G$  has an induced  $K_l$  or an induced  $I_k$ .

Case 2: Suppose  $G^C$  has an induced  $I_l$ . Then  $G$  has an induced  $K_l$ . Thus  $G$  has an induced  $K_l$  or an induced  $I_k$ .

So,  $f(l, k) \leq f(k, l)$ . Thus,  $f(k, l) = f(l, k)$ .  $\square$

**Lemma 2.9.** *Let  $l \in \mathbb{N}$ . Then  $h(3, l) = f(3, l)$ .*

Proof. Let  $l \in \mathbb{N}$ . Let  $G$  be a graph. Observe that a complete graph of order 3 is the same as a cycle of order 3. Thus,  $G$  is guaranteed to have an a  $C_3$  subgraph or an induced  $I_l$  if and only if  $G$  is guaranteed to have an induced  $K_3$  or induced  $I_l$ .  $\square$

### 2.3 Results of $h(k, l)$

In this section, we focus on Ramsey-like numbers. In our case, this means we looked for the number of vertices required to guarantee a cycle subgraph of order  $k$  or an induced complete graph of order  $l$ . We decompose the problem of finding Ramsey numbers into smaller manageable pieces. Mathematicians and effective problem solvers use this strategy often. Sometimes after solving enough pieces, a solution for the original problem appears.

Notice how we break the main claim of this section,  $h(k + 1, 3) = 2k + 1$ , into three separate parts (theorem 2.10, lemma 2.11, and theorem 2.12).

Let us begin to inspect and understand the pieces of our main claim. We begin by establishing a lower bound for the values  $h(k, l)$ .

**Theorem 2.10.** *Let  $k, l \in \mathbb{N}$ . Then  $h(k + 1, l + 1) > kl$ .*

*Proof.* We will prove that  $h(k + 1, l + 1) > kl$  by providing a counterexample. Let  $K_1, K_2, \dots, K_l$  be complete graphs of order  $k$  such that  $K_1, K_2, \dots, K_l$  are pairwise disjoint. Consider  $G = \bigcup_{i=1}^l K_i$ . We will show that  $G$  does not have a  $C_{k+1}$  subgraph nor an induced  $I_{l+1}$ .

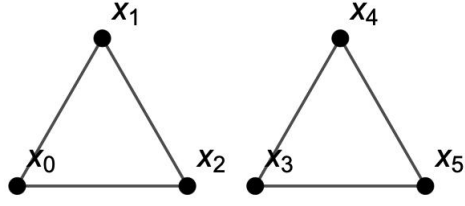
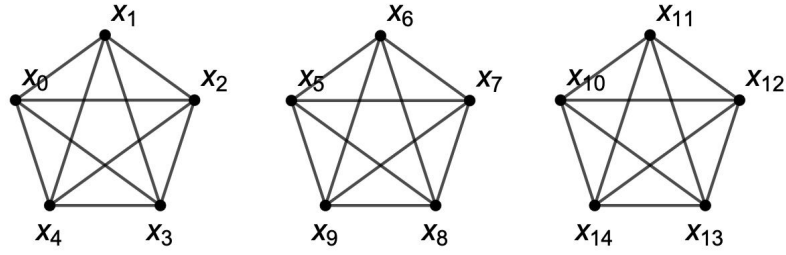
Case 1: Suppose  $G$  has a  $C_{k+1}$  subgraph. Since  $G = \bigcup_{i=1}^l K_i$  and  $K_1, K_2, \dots, K_l$  are pairwise disjoint,  $V(C_{k+1}) \subset V(K_i)$  for some  $i$  such that  $1 \leq i \leq l$ . Without loss of generality, let  $V(C_{k+1}) \subset V(K_1)$ . Then  $k + 1 = |V(C_{k+1})| \leq |V(K_1)| = k$ .  $\nexists$

Case 2: Suppose  $G$  has an induced  $I_{l+1}$ . Since  $G = \bigcup_{i=1}^l K_i$  and  $K_1, K_2, \dots, K_l$  are all complete,  $V(I_{l+1})$  includes at most one vertex from  $K_i$  for all  $i \leq l$ . Thus  $|V(I_{l+1})| \leq l$ . However,  $|V(I_{l+1})| = l + 1$ .  $\nexists$

Thus  $G$  has neither a  $C_{k+1}$  subgraph nor an induced  $I_{l+1}$ . □

In fig. 2.4 and ??, we show what the graph  $G$  from theorem 2.10 would look like for two different  $k, l$  pairs.



Figure 2.4:  $h(4, 3)$  counterexampleFigure 2.5:  $h(6, 4)$  counterexample

If an additional vertex is invited into a cycle of length  $k$ , then a new cycle of length  $k + 1$  is formed. This fact is used frequently in theorem 2.12.

**Lemma 2.11** (Adjacency Lemma). *Let  $x_0$  and  $x_1$  be two adjacent vertices of a  $C_k$ . Suppose  $x_0$  and  $x_1$  are both adjacent to some vertex  $y \notin V(C_k)$ . Then,*

$$x_0 - y - x_1 - x_2 - \cdots - x_{k-1} - x_0$$

*forms a  $C_{k+1}$ .*

The proof of lemma 2.11 is omitted; the lemma is clear by a simple observation.

We are about to prove the main result of this thesis; we determine a whole class of  $h(k, l)$  values (when  $l = 3$ ). The main tactics used in this proof can be found in lemma 2.6

and lemma 2.11. After the base case is provided, we take a somewhat constructive approach to proving the induction.

**Theorem 2.12.** *Let  $k \in \mathbb{N}$  such that  $k \geq 3$ . Then  $h(k+1, 3) = 2k+1$ .*

Proof. Let  $k \in \mathbb{N}$ . We will prove  $h(k+1, 3) = 2k+1$  for  $k \geq 3$  by induction on  $k$ .

Base Case: Let  $k = 3$ . We will show  $h(4, 3) = 7$ . Let  $G$  be a graph such that  $|V(G)| = 7$ . Suppose that  $G$  does not have an induced  $I_3$ . We will show  $G$  has a  $C_4$  subgraph.

By lemma 2.2, we know that  $f(3, 3) = 6$ . Since  $|V(G)| = 7$ ,  $h(3, 3) = f(3, 3) = 6$ , and  $G$  does not have an induced  $I_3$ ,  $G$  has a  $C_3$  subgraph. Let  $V(C_3) = \{x_0, x_1, x_2\}$ . Let  $H$  be the induced subgraph of  $G$  such that  $V(H) = V(G) \setminus V(C_3)$ . Notice  $|V(H)| = 7 - 3 = 4$ . Let  $V(H) = \{y_0, y_1, y_2, y_3\}$ .

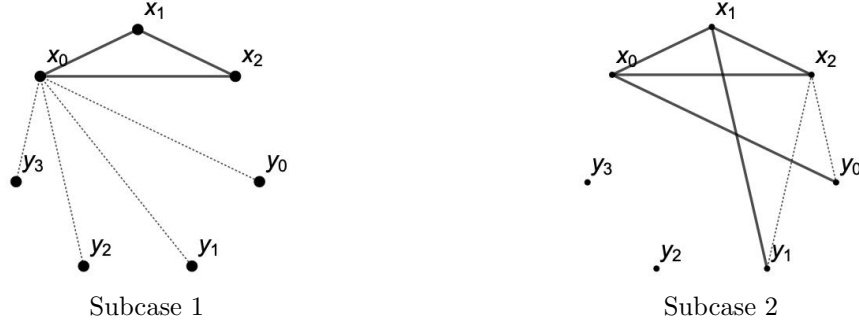


Figure 2.6

Subcase 1: Suppose some element in  $V(C_3)$ , namely  $x_0$ , is not adjacent to any vertex in  $V(H)$ . So,  $x_0$  is not adjacent to  $y_i$  for  $i \in \{0, 1, 2, 3\}$  (see fig. 2.6 Subcase 1). Since  $G$  does not have an induced  $I_3$ , for  $i \neq j$  with  $i, j \in \{0, 1, 2, 3\}$ ,  $y_i$  and  $y_j$  are adjacent in  $G$  and thus in  $H$ . Thus, subgraph  $H$  is a complete graph of order 4. Thus,  $H$  has a  $C_4$  subgraph. Hence,  $G$  has a  $C_4$  subgraph.

Subcase 2: Suppose that every element in  $V(C_3)$  is adjacent to some element in  $V(H)$ . By lemma 2.11, we suppose that no two adjacent vertices of  $C_3$  are adjacent to the same vertex of  $H$ .

So, let  $x_0$  be adjacent to  $y_0$  and  $x_1$  be adjacent to  $y_1$  (see fig. 2.6 Subcase 2). Note in fig. 2.6 Subcase 2, we omit the edge that joins  $x_2$  with some vertex of  $H$  as it does not aid in the proof of the theorem. Notice,  $x_2$  is neither adjacent to  $y_0$  nor  $y_1$  by lemma 2.11. Since  $x_2$  is not adjacent with  $y_0$  nor  $y_1$  and  $G$  does not have an induced  $I_3$ ,  $y_0$  and  $y_1$  are adjacent. Observe,  $\{x_0, x_1, y_1, y_0\}$  are the vertices of a  $C_4$ . Thus  $G$  has a  $C_4$  subgraph.

Thus  $h(4, 3) \leq 7$ . From theorem 2.10, we know  $h(4, 3) > 6$ . Using both of these facts, we have  $h(4, 3) = 7$ .

Let us proceed with our induction on  $k$ . Suppose that  $h(k+1, 3) = 2k+1$ . We will show that  $h(k+2, 3) = h((k+1)+1, 3) = 2(k+1)+1 = 2k+3$ . Let  $G$  be a graph such that  $|V(G)| = 2k+3$ . Suppose that  $G$  does not have an induced  $I_3$ . We will show  $G$  has a  $C_{k+2}$  subgraph.

Since  $|V(G)| = 2k+3$ ,  $h(k+1, 3) = 2k+1$ , and  $G$  does not have an induced  $I_3$ ,  $G$  has a  $C_{k+1}$  subgraph. Let  $V(C_{k+1}) = \{x_0, x_1, \dots, x_k\}$ . Let  $H$  be the induced subgraph of  $G$  such that  $V(H) = V(G) \setminus V(C_{k+1})$ . Notice  $|V(H)| = k+2$ . Let  $V(H) = \{y_0, y_1, \dots, y_{k+1}\}$ .

Subcase 1:

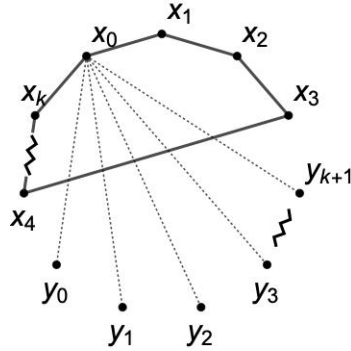


Figure 2.7

Suppose some element in  $V(C_{k+1})$ , namely  $x_0$ , is not adjacent to any vertex in  $V(H)$ . So,  $x_0$  is not adjacent to  $y_i$  for  $i \in \{0, \dots, k+1\}$  (see fig. 2.7). Since  $G$  does not have an induced  $I_3$ , for  $i \neq j$  with  $i, j \in \{0, \dots, k+1\}$ ,  $y_i$  and  $y_j$  are adjacent in  $G$  and thus in  $H$ . Thus, subgraph  $H$  is a complete graph of order  $k+2$ . Thus,  $H$  has a  $C_{k+2}$  subgraph. Hence,  $G$  has a  $C_{k+2}$  subgraph.

Subcase 2: Suppose that every element in  $V(C_{k+1})$  is adjacent to some element in  $V(H)$ . By lemma 2.11, we suppose that no two adjacent vertices of a  $C_{k+1}$ , disjoint from  $H$ , are adjacent to the same vertex of  $H$ . Let  $x_0$  and  $x_1$  be adjacent to  $y_0$  and  $y_1$ , respectively.

Note in fig. 2.8; fig. 2.9; fig. 2.10; and fig. 2.11, we omit the edges that join  $x_i$  for  $i \in \{3, 4, \dots, k\}$  with some vertex of  $H$  as they do not aid in the proof of the theorem. Also, we later discern the edge that joins  $x_2$  with its vertex in  $V(H)$  as shown in fig. 2.10.

Since no two adjacent vertices of

$$x_0 - x_1 - x_2 - \dots - x_k - x_0 \quad (2.1)$$

are adjacent to the same vertex in  $V(H)$ ,  $x_0$  and  $y_1$  are not adjacent. Similarly,  $x_2$  and  $y_1$  are not adjacent (see fig. 2.8 (a)).

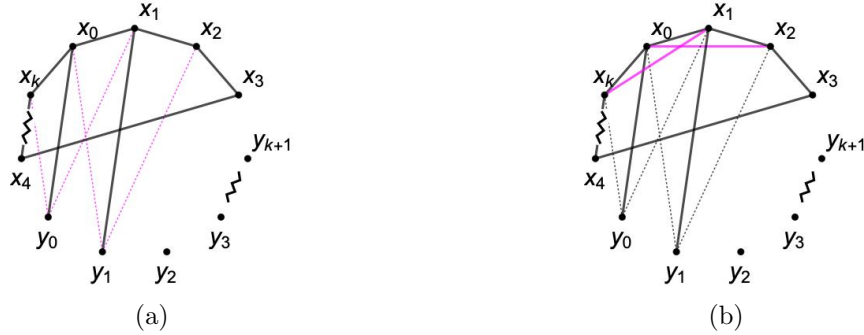


Figure 2.8

Since  $x_0$  and  $x_2$  are not adjacent to  $y_1$  and  $G$  does not have an induced  $I_3$ ,  $x_0$  and  $x_2$  are adjacent. Using no two adjacent vertices of a  $C_{k+1}$  are adjacent to the same vertex in  $V(H)$ , by a similar argument,  $x_k$  and  $x_1$  are adjacent (see fig. 2.8 (b)).

Notice,

$$x_0 - x_2 - x_3 - x_4 - \cdots - x_{k-1} - x_k - x_1 - x_0 \quad (2.2)$$

is a different  $C_{k+1}$  than the one in eq. (2.1) that is also disjoint from  $H$ .

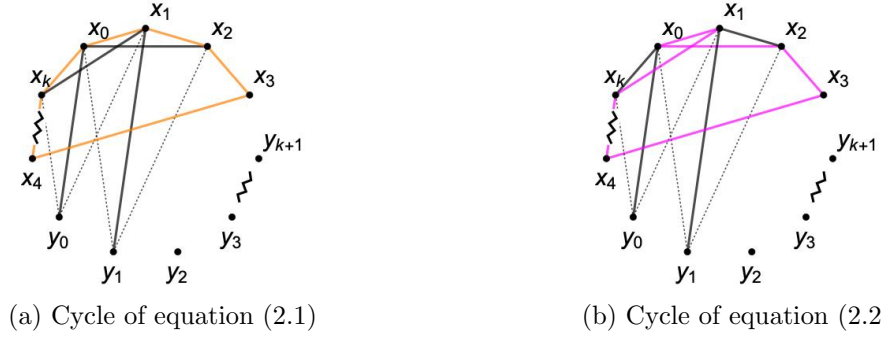


Figure 2.9

In fig. 2.9 (a) and (b), we draw attention to the two unique cycles of equation 2.1 and equation 2.2, respectively. In the  $C_{k+1}$  of equation 2.2,  $x_0$  and  $x_2$  are adjacent. Thus,  $x_2$  is not adjacent to  $y_0$  (see fig. 2.10 (a)). Since  $x_2$  is adjacent to both  $x_0$  and  $x_1$ ,  $x_2$  is not adjacent to  $y_0$  nor  $y_1$  by lemma 2.11. Let  $x_2$  be adjacent to  $y_2$  as in fig. 2.10 (b).

Now that  $x_2$  is not adjacent to  $y_0$  nor  $y_1$ , since  $G$  does not have an induced  $I_3$ ,  $y_0$  and  $y_1$  are adjacent (see fig. 2.10 (b)).

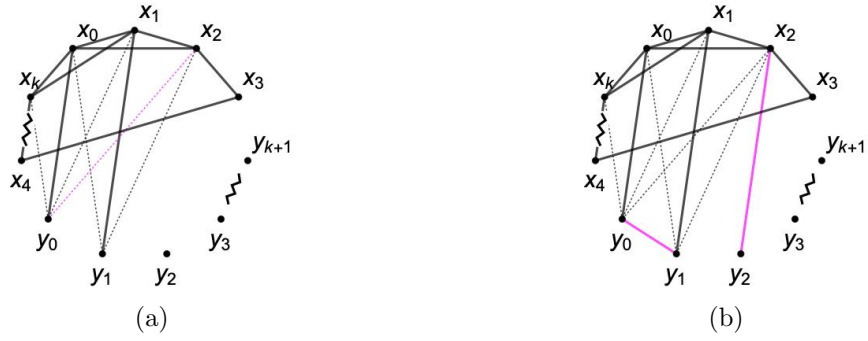


Figure 2.10

Again, since no two adjacent vertices of a  $C_{k+1}$  are adjacent to the same vertex in  $V(H)$ ,  $x_1$  and  $y_2$  are not adjacent. Similarly,  $x_3$  and  $y_2$  are not adjacent (see fig. 2.11 (a)). Since  $G$  does not have an induced  $I_3$ ,  $x_1$  and  $x_3$  are adjacent (see fig. 2.11 (b)).

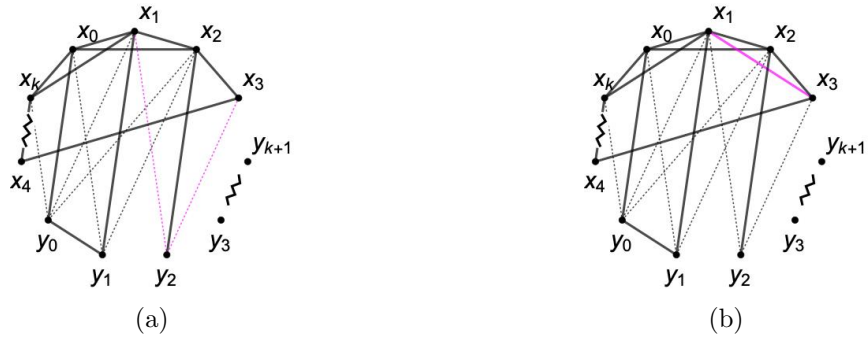


Figure 2.11

Notice,

$$x_0 - y_0 - y_1 - x_1 - x_3 - x_4 - \dots - x_{k-1} - x_k - x_0 \quad (2.3)$$

is a  $C_{k+2}$ . The cycle mentioned in equation 2.3 is shown in fig. 2.12.

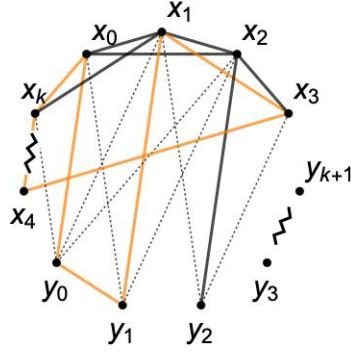


Figure 2.12

Thus  $G$  has a  $C_{k+2}$  subgraph.

Thus  $h(k+2, 3) \leq 2k+3$ . From theorem 2.10, we know

$h(k+2, 3) > (k+1) \cdot 2 = 2k+2$ . Using both of these facts, we have  $h(k+2, 3) = 2k+3$ .  $\square$

We already know by lemma 2.6,  $h(k, 3) \leq f(k, 3)$ , but how much more is  $f(k, 3)$  than  $h(k, 3)$ ? Now that we know the value for  $h(k, 3)$  for  $k \geq 3$ , we have more information to answer such a question as how  $h(k, l)$  and  $f(k, l)$  are related. Perhaps, by taking  $h(k, 3)$  up one notch to  $h(k+1, 3)$ , requiring more, we can get a closer lower bound for  $f(k, 3)$ .

We work to uncover this relationship by proving theorem 2.13.

**Theorem 2.13.** *Let  $k \in \mathbb{N}$  such that  $k \geq 4$ . Then,  $f(k, 3) > 2k$ .*

*Proof.* Let  $k \in \mathbb{N}$  such that  $k \geq 4$ . We shall proceed by induction on  $k$ .

Base Case: Let  $k = 4$ . It is known that  $f(4, 3) = 9$ . Thus,  $f(4, 3) = 9 > 8 = 2(4)$ .

Let  $f(k, 3) > 2k$ . We will show  $f(k+1, 3) > 2(k+1)$ . Since  $f(k, 3) > 2k$ , let  $G$  be a graph such that  $|V(G)| = 2k$  and  $G$  fails to have both an induced  $K_k$  and an induced  $I_3$ . Let  $V(G) = \{x_0, x_1, \dots, x_{2k-1}\}$ . Let  $G'$  be a graph such that  $G$  is an induced subgraph of  $G'$  and  $V(G') = V(G) \cup \{y_0, y_1\}$  such that  $y_0$  and  $y_1$  are not adjacent and  $y_i$  and  $x_j$  are adjacent for all  $i \in \{0, 1, \dots, 2k-1\}$  and  $j \in \{0, 1\}$ .

Notice,  $y_0$  and  $y_1$  do not participate in an  $I_3$  subgraph of  $G'$ . Since  $y_0$  and  $y_1$  do not participate in an  $I_3$  subgraph of  $G'$  and  $G$  does not have an induced  $I_3$ ,  $G'$  does not

have an induced  $I_3$ .

Using a contradiction argument, we will now show that  $G'$  does not have an induced  $K_{k+1}$ . Suppose  $G'$  has an induced  $K_{k+1}$ . Since  $y_0$  and  $y_1$  are not adjacent, the  $K_{k+1}$  of  $G'$  does not contain both  $y_0$  and  $y_1$ . However, since  $G$  does not have an induced  $K_k$ ,  $y_0$  or  $y_1$  must be in the  $K_{k+1}$ .

Without loss of generality, suppose  $y_0 \in V(K_{k+1})$ . Then,  $V(K_{k+1}) \setminus \{y_0\}$  are the vertices of a  $K_k$ . Notice  $V(K_{k+1}) \setminus \{y_0\} \subset V(G)$ . Thus  $G$  has an induced  $K_k$ .  $\nexists$

So  $G'$  fails to have an induced  $K_{k+1}$ . Hence,  $G'$  is a graph such that  $|V(G')| = 2k + 2$  and  $G'$  fails to have an induced  $K_{k+1}$  and an induced  $I_3$ .

Thus  $f(k + 1, 3) > 2(k + 1)$ . □

**Corollary 2.14.** *Let  $k \in \mathbb{N}$  such that  $k \geq 4$ . Then  $h(k + 1, 3) \leq f(k, 3)$ .*

Proof. Let  $k \in \mathbb{N}$  such that  $k \geq 4$ . By theorem 2.13,  $f(k, 3) > 2k$ . So,  $f(k, 3) \geq 2k + 1$ . By theorem 2.12,  $h(k + 1, 3) = 2k + 1 \leq f(k, 3)$ . □



## CHAPTER 3

### Forward

#### 3.1 Additional Work

Now that we have found the value of  $h(k, 3)$  for  $k \geq 3$ , we attempt to extend our knowledge and find values for  $h(k, l)$  for  $l \geq 4$ . We start this discussion by first realizing that  $h(k, l)$  does not maintain a property that  $f(k, l)$  holds. Specifically,  $h(k, l) \neq h(l, k)$  in general.

**Lemma 3.1.** *There exists  $k, l \in \mathbb{N}$  such that  $h(k, l) \neq h(l, k)$ .*

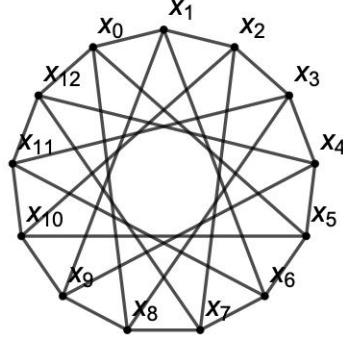
Proof. Consider  $h(4, 3)$  and  $h(3, 4)$ . By theorem 2.12, we know  $h(4, 3) = 7$ . By lemma 2.9 and lemma 2.4, we know  $h(3, 4) = f(3, 4) = 9$ . Thus,

$$h(4, 3) = 7 \neq 9 = h(3, 4).$$

□

In pursuit of determining new  $h(k, l)$  values, we conclude this thesis with a counterexample for  $h(3, 5)$ . Thus, we have found a lower bound for  $h(3, 5)$ . It is also worth mentioning by lemma 2.9, this counterexample is also a counterexample for  $f(3, 5)$ .

**Example 3.2.** *The graph in fig. 3.1 below is an example of a graph that does not have a  $C_3$  subgraph nor an induced  $I_5$ .*

Figure 3.1:  $h(3, 5), f(3, 5)$  counterexample

Proof. Let  $G$  be the graph as in fig. 3.1. We first characterize the graph  $G$ . Then we prove that  $G$  fails to have a  $C_3$  subgraph. Finally, we will show that  $G$  does not have an induced  $I_5$ .

If arithmetic is done in  $\mathbb{Z}_{13}$ ,  $x_i$  is adjacent to  $x_{i-1}, x_{i+1}, x_{i-5}$ , and  $x_{i+5}$  for  $i \in \{0, 1, \dots, 12\}$ .

By way of contradiction, suppose  $G$  has a cycle subgraph of order 3 called  $C_3$ . Without loss of generality, let  $x_0 \in V(C_3)$ .

Let  $x_j, x_k \in V(C_3)$  for some  $j, k \in \{1, 2, \dots, 12\}$  with  $j, k \neq 0$ . Since  $x_0 \in C_3$ ,  $x_0$  is adjacent to  $x_j$  and  $x_k$ . Similarly,  $x_j$  is adjacent to  $x_k$ . Since  $x_j$  and  $x_k$  are adjacent to  $x_0$ ,  $j, k \in \{1, 5, 8, 12\}$ . However, notice  $x_1, x_5, x_8$ , and  $x_{12}$  are all pairwise disjoint. Thus  $x_j$  and  $x_k$  are not adjacent.  $\nexists$

Thus  $G$  does not have a cycle subgraph of order 3.

We now prove that  $G$  does not have an induced  $I_5$ . By way of contradiction, suppose  $G$  has an induced  $I_5$ . Since each vertex is indistinguishable, without loss of generality, let  $x_0 \in I_5$ .

Then,  $x_1, x_5, x_8, x_{12} \notin I_5$ . We partition the remaining vertices as such:  $\{x_2, x_3, x_4\}$ ,  $\{x_6, x_7\}$ , and  $\{x_9, x_{10}, x_{11}\}$ . Since we have four more points in our  $I_5$ , and there are three groups in our partition, by the pigeonhole principle, at least two vertices of the  $I_5$  will

come from the same set of vertices.

Two vertices of the  $I_5$  cannot be elements of  $\{x_6, x_7\}$  as  $x_6$  and  $x_7$  are adjacent.

Since  $\{x_2, x_3, x_4\}$  and  $\{x_9, x_{10}, x_{11}\}$  are indistinguishable, without loss of generality, suppose at least two vertices of the  $I_5$  are elements of  $\{x_2, x_3, x_4\}$ . Notice there is only one way to select at least two vertices from  $\{x_2, x_3, x_4\}$  where the selected vertices are pairwise disjoint. Thus,  $x_2, x_4 \in I_5$ . Hence  $x_3, x_7, x_9, x_{10} \notin I_5$ .

Thus the only other vertices available for the  $I_5$  are  $x_6$  and  $x_{11}$ . So  $x_6, x_{11} \in I_5$ . However,  $x_6$  and  $x_{11}$  are adjacent.  $\nexists$

Thus  $G$  does not have an induced  $I_5$ .

We have shown that  $G$  does not have a  $C_3$  subgraph nor an induced  $I_5$ . Thus,  $f(3, 5) \geq 14$ .

□

What is in store next? In the next section, we aim to give a brief summary of where the field of Ramsey Theory currently is, and what questions are still open.

### 3.2 Future Work

Now, we were not the first ones to pose Ramsey-like questions, and we certainly will not be the last. Our main result was actually first found in 1971 [ChaS]. When it comes to the function  $h(k, l)$ , many more values are known. In fact, if we were to make a new discovery, we would have to uncover  $h(k, 8)$  for  $k \geq 10$ . This value is conjectured to be  $7k - 6$ . A table of known  $h(k, l)$  values can be found in table 3.1 [30].

Some notes on table 3.1 in regards to keeping consistent with Radziszowski [30]: citations with "-" describe joint credit; the first reference is for the lower bound and the second the upper bound. As well, "/" denotes joint credit. Some papers contain results found with the help of computer algorithms. A list of papers that use some computers where results are easily verifiable with some computations are: [9],[10],[11],[26],[32]. A list of papers where cpu intensive algorithms have to be used to verify or replicate the results are: [1],[12], [16], [29].

Table 3.1: Known  $h(k, l)$  values denoted  $f(C_k, K_l)$ 

	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$	$\dots C_k$ for $k \geq l$
$K_3$	6	7	9	11	13	15	17	$\dots 2k - 1$
	[15]-[4]	[5]	$\dots$					$\dots [5]$
$K_4$	9	10	13	16	19	22	25	$\dots 3k - 2$
	[15]	[6]	[18]/[25]	[23]	[40]	$\dots$		$\dots [40]$
$K_5$	14	14	17	21	25	29	33	$\dots 4k - 3$
	[15]	[7]	[17]/[25]	[23]	[41]	[2]	$\dots$	$\dots [2]$
$K_6$	18	18	21	26	31	36	41	$\dots 5k - 4$
	[27]	[9]-[34]	[24]	[35]	$\dots$			$\dots [35]$
$K_7$	23	22	25	31	37	43	49	$\dots 6k - 5$
	[26]-[14]	[32]-[22]	[36]	[37]	[37]	[21]/[39]	[39]	$\dots [39]$
$K_8$	28	26	29-33	36	43	50	57	$\dots 7k - 6$
	[16]-[29]	[32]	[20]	[38]	[39]	[19]/[42]	[28]	$\dots$ conj.
$K_9$	36	30						$\dots 8k - 7$
	[26]-[16]	[32]-[1]						$\dots$ conj.
$K_{10}$	40-42	36						$\dots 9k - 8$
	[10]-[12]	[1]						$\dots$ conj.
$K_{11}$	47-50	39-44						$\dots 10k - 9$
	[11]-[12]	[1]						$\dots$ conj.

Interestingly enough, there are other Ramsey-like questions that can also be explored; we focused on cycles, but we could have looked at almost complete graphs versus almost complete graphs, cycles versus cycles, cycles versus stars, cycles versus wheels, cycles versus books, etc. Note that almost complete graphs, stars, and wheels are all well defined graphs with examples shown in fig. 3.2; fig. 3.3; and fig. 3.4, respectively.

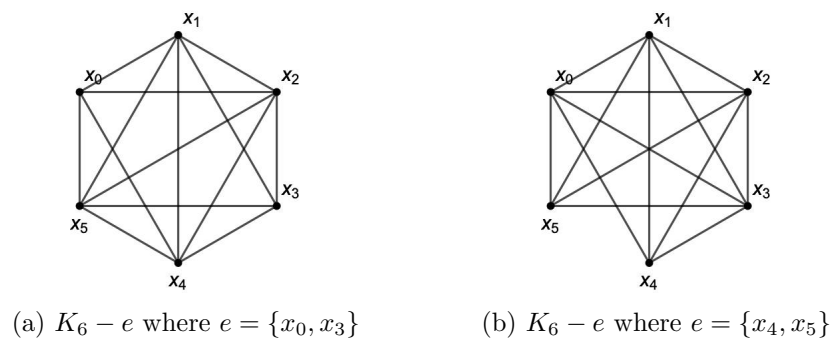


Figure 3.2

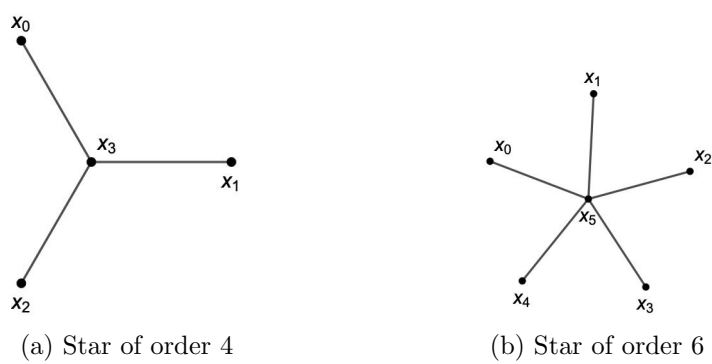
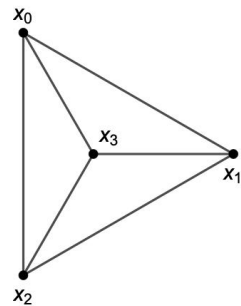
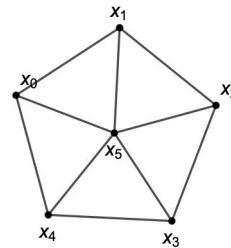


Figure 3.3



(a) Wheel of order 4



(b) Wheel of order 6

Figure 3.4

Besides looking into other types of induced subgraphs with different structure, we can also look into multicolor Ramsey numbers. In this thesis, edges of graphs either existed or did not exist; hence, all proofs worked with Ramsey numbers of two colors. For reference, twenty three Ramsey numbers are known for the three color case, and no exact Ramsey numbers are known for the four color case. Then, again, we could look into multicolor Ramsey-like numbers that correspond to guaranteeing cycles, stars, wheels, etc.

If looking to learn more about Ramsey's Theorem and its related questions, a great place to start would be the Mathematical Review completed by Stanisław P. Radziszowski [30]. Radziszowski has cited over 700 references. From the review and cited papers, in depth knowledge on the topic can be collected.

If looking to get fairly young students interested in the subject, I recommend looking at Ramsey Theory [13]. It introduces the topic in an interesting and friendly way with plenty of visuals for the reader to investigate.

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